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SOME NONLINEARITIES IN REGULATORY SYSTEMS

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/Figures are appended./

Every regulated system is essentially a nonlinear system; the reason is that even if the regulated object and the measuring element of the regulator can, in the first approximation, be considered as linear systems, still the power-amplifying element of the regulator will always be nonlinear, in view of the large values of amplification as contrasted with the limited power of the regulation element. The nature of this nonlinearity can be perceived in the first approximation as shown in Figure 1.

The abscissa axis gives the input values of the power-amplifying element and the ordinate axis, the output. In the range of variation between -a and a, the system is obviously linear. In regulators operating with large amplifications (small "staticism"), the value of a can be less than 0.005-0.01. During any disturbance of a regulated system the deflection x_1 is generally much greater than a; therefore, a regulated system in practice always behaves nonlinearally.

The presence of friction, free play, and insensitivity zones in regulators makes the problem of regulation substantially even more nonlinear. To study a regulation system as a linear approximation can lead to incorrect results. With dry friction not taken into consideration, analysis can show a given linear regulation system to be stable even if, when dry friction is taken into account, oscillations can be shown to arise for known conditions in a system whose linear part is stable /1, 2, 3/.

The method presented below for solving the problem of stationary movements in a nonlinear regulation system is just the first approximation in the problem's solution and is based on the principle of harmonic equilibrium (balance) /4/.

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In essence the method is analogous to Van der Pol's method [5]. In many cases, the oscillations in regulatory systems are of a quasi-harmonic nature; therefore the first approximation of the solution in practice gives fully satisfactory results [2, 6].

The Concept of Equivalent Admittance of a Nonlinear Element

Let a certain linear element of a regulation system be characterized by a relation between the "output" x_2 and the "input" x_1 in the form of an operational equation:

$$x_2 = J(p)x_1 \tag{1}$$

The admittance of the linear element or its amplification factor (coefficient) is the name for the ratio of output to input (here and in the rest of the text it is understood that the parameters of the system are dimensionless (reduced) relative quantities):

$$x_2/x_1 = J(p) \tag{2}$$

A basic peculiarity of the linear element is the absence of any dependence between the element's admittance and the input signal.

In a particular case $J(p)$ can be equal to a constant C. In this case x_2 repeats without distortion in time (but on a different scale) the quantity x_1.

Let there be given a nonlinear element which connects output x_2 and input x_1 by the function

$$x_2 = \phi(x_1) \tag{3}$$

then the expression for equivalent admittance will have the form [4]:

$$J_{no} = g_{ek}(A_1) + jb_{ek}(A_1) \tag{4}$$

where

$$g_{ek}(A_1) = \frac{1}{\pi A_1}\int_0^{2\pi}\phi(A_1 \sin\psi)\sin\psi\, d\psi \tag{5a}$$

and

$$b_{ek}(A_1) = \frac{1}{\pi A_1}\int_0^{2\pi}\phi(A_1 \sin\psi)\cos\psi\, d\psi \tag{5b}$$

$$x_1 = A_1 \cdot \sin\psi$$

In this case the equivalent admittance of the nonlinear element depends upon the amplitude A_1 of the input signal. If the function phi(x_1) is a single-valued (unique) odd function, then the reactive admittance of the equivalent admittance is $b_{ek} = 0$ and hence the expression for equivalent admittance takes the simple form: $J_{ns} = g_{ek}(A_1)$. Analyzing expression (5), we come to the conclusion that the equivalent admittance of the nonlinear element is determined by the ratio of the complex vector of the basic harmonic at output to the vector of the same harmonic at input; hence it is assumed that at input only one harmonic oscillation of arbitrary frequency is possible.

Equivalent Admittances of Some Nonlinear Elements of Regulated Systems

Let us examine a number of nonlinear elements often encountered in regulation systems and determine their equivalent admittances.

1. Zone of Insensitivity of the First Type

The characteristic of this type of nonlinear element (NEI) is shown in Figure 2a.

In Figure 2b is shown an elementary model having a similar characteristic. Shaft I is connected with shaft II through a fork of width 2a. On shaft II is mounted a spring which tends to place the shaft in a zero (neutral) position. If the angle of rotation of shaft I is taken for x_1, then the angle of rotation of shaft II, namely x_2, will depend upon x_1 according to the characteristic curve in Figure 2a.

The equivalent admittance NEI, according to the foregoing, will consist only of the active conductance.

For convenience in future calculations, we will define the equivalent admittance as some function of the ratio of amplitude A_1 of oscillation of the input quantity x_1 to half the width 2a of the insensitivity zone, namely a.

According to (4) and (5a) we get:

$$J_{NEI0} = g_{ek}\left(\frac{A_1}{a}\right) = \frac{4}{\pi A_1}\int_{\alpha}^{\pi/2}(A_1 \sin\psi - a)\sin\psi \, d\psi = 1 - \frac{2\alpha}{\pi} + \frac{1}{\pi}\sin 2\alpha - \frac{4}{\pi \frac{A_1}{a}}\cos\alpha \qquad (6)$$

where alpha equals arc sin a/A_1.

In Figure 3 is shown the dependence of the equivalent admittance J_{NEI_0} upon the ratio A_1/a. The value of J_{NEI_0} tends to the limit 1 as the ratio A_1/a tends to infinity. If the absolute value of dx_2/dx_1 on the rectilinear part of the characteristic of $\varphi(x_1)$ differs from 1 and equals N, then the equivalent admittance of a similar element will equal NJ_{NEI_0}.

2. Zone of Insensitivity of the Second Type

The characteristic of this type of a nonlinear element (NEI) is shown in Figure 4a.

In Figure 4b is shown an elementary model having a similar characteristic. During the motion of a sliding contact I, determined by the coordinate x_1, the motor M is not switched on in the interval 2a of the fork (2a is the distance between the static contacts II). For absolute x_1 greater than a, a constant voltage equal to the absolute value of $x_2 = B$ is fed to the motor. According to (4) and (5a) we get:

$$J_{NEII} = g_{ek}\left(\frac{A_1}{a}\right) = \frac{4}{\pi A_1}\int_{\alpha}^{\pi/2} B \sin\psi \, d\psi = N \cdot J_{NEII_0} \qquad (7)$$

where

$$\alpha = \text{arc} \sin a/A_1, \qquad (8)$$

and

$$J_{NEII_0} = \frac{4}{\pi}\cdot\frac{1}{A_1/a}\sqrt{1 - \frac{1}{(A_1/a)^2}}$$

$$N = B/a$$

In Figure 5 is shown the dependence of the equivalent admittance J_{NEII0} upon the ratio A_1/a, according to formula (8). The maximum value of J_{NEII0} is equal to 0.637 and holds true for A_1/a equal to 1.41.

3. Nonlinear Element With Free Play or Friction (NEIII)

Let free play exist in one of the links of the regulator's mechanism. An elementary scheme imitating free play is shown in Figure 6b. If the angle of rotation of shaft I, measured by coordinate x_1, is within the interval 2a of the width of the fork, then the output shaft II remains stationary. For absolute x_1 greater than a, shaft II follows shaft I. The characteristic of the similar nonlinear element is shown in Figure 6a. This characteristic is double-valued and is denoted by a loop.

Let us examine the nonlinearity connected with the presence of dry friction. An elementary scheme imitating dry friction is shown in Figure 6c. On shaft I acts a moment determined by coordinate x_1. The shaft is held in block k, which creates dry friction; the moment of friction equals a. For absolute x_1 greater than a the angle of rotation of the shaft, which is determined by coordinate x_2, is equal to zero. The equations connecting x_2 and x_1 can be written in the following form:

$$\begin{aligned} x_2 &= N(x_1 - a) && \text{for } \dot{x}_2 > 0 \\ x_2 &= N(x_1 + a) && \text{for } \dot{x}_2 < 0 \\ |x_2 - Nx_1| &< Na && \text{for } \dot{x}_2 = 0 \end{aligned} \tag{9}$$

where N is a coefficient determined by the inflexibility of the spring

The graphic illustration of the relation between x_2 and x_1 will have the same form as in Figure 6a, if we take N = 1. Thus free play and dry friction have analogous characteristics.

Equivalent admittance of a nonlinear element with dry friction (for N = 1) or free play (NEIII), according to equations (4) and (5), will be equal to

$$J_{NEIII_0} = g_{ek}\left(\frac{A_1}{a}\right) + j b_{ek}\left(\frac{A_1}{a}\right) \tag{10}$$

where

$$g_{ek}\left(\frac{A_1}{a}\right) = \frac{2}{\pi A_1}\int_{\alpha}^{\pi} \phi(A_1 \sin\psi)\cdot\sin\psi\cdot d\psi = \frac{2}{\pi}\left[\frac{3}{4}\pi - \frac{\alpha}{2} - \cos\alpha + \frac{1}{4}\sin 2\alpha + \frac{a}{A_1}\cdot 2\cos\alpha\right] \tag{10a}$$

$$b_{ek}\left(\frac{A_1}{a}\right) = \frac{2}{\pi A_1}\int_{\alpha}^{\pi} \phi(A_1 \sin\psi)\cdot\cos\psi\cdot d\psi = \frac{2}{\pi}\left[-\frac{3}{4} + \sin^2\alpha + \frac{1}{4}\cdot\cos 2\alpha\right]$$

$$\alpha = \text{arc sin}\left[-\frac{2a}{A_1}\right] \tag{10b}$$

Equation (10) can be rewritten in the form

$$J_{NEIII_0} = \left|J_{NEIII_0}(A_1/a)\right|\cdot \exp\, j\gamma(A_1/a)$$

where

$$\gamma\cdot(A_1/a) = \text{arc tan}\; b_{ek}/g_{ek}$$

and

$$\left|J_{NEIII_0}(A_1/a)\right| = \sqrt{g_{ek}^2 + b_{ek}^2} \tag{11}$$

Figure 7 shows the variation in output x_2 for a sinusoidal variation in x_1. The graph is plotted in accordance with equation (9), for $N = 1$ (Figure 6a). The value of J_{NEII0} according to equation (10) is defined as the ratio of the amplitude of the first harmonic of x_2 to the amplitude of the harmonic at input A_1. Angle gamma determines the phase displacement between the first harmonic of x_2 at output of the nonlinear element and the same harmonic at input. The value of gamma is less than zero; this means that the first harmonic at output lags behind (as to phase) the harmonic at input. In Figure 8 is plotted active conductance g_{ek} and reactive conductance b_{ek} versus the ratio A_1/a.

Figure 9 gives the curves of coefficient J_{NEII0} and phase gamma versus the ratio A_1/a. The modulus varies from 0 (for $A_1/a = 1$) to 1 (for A_1/a = infinity); the phase varies from 90 degrees (for $A_1/a = 1$) to 0 (for A_1/a = infinity). In future calculations it will be necessary to make use of the curve giving the impedances of a nonlinear element; that is, the locus (curve) of the geometric positions of the vector termini of $1/J_{NEIII_0}$.

The quantity $1/J_{NEIII_0}$ will equal:

$$\frac{1}{J_{NEIII0}}\left(\frac{A_1}{a}\right) = \frac{1}{g_{ek} + j b_{ek}} = \frac{g_{ek}}{g_{ek}^2 + b_{ek}^2} - j\frac{b_{ek}}{g_{ek}^2 + b_{ek}^2} = X(A_1/a) + jY(A_1/a), \tag{12}$$

where

$$X(A_1/a) = g_{ek}/(g_{ek}^2 + b_{ek}^2)$$

and

$$Y(A_1/a) = -b_{ek}/(g_{ek}^2 + b_{ek}^2)$$

The curve tracing the terminus of the vector $1/J_{NEIII_0}$ (A_1/a) is plotted in Figure 10. On the abscissa axis is plotted the value of $x(A_1/a)$ and on the ordinate axis, $Y(A_1/a)$.

On the curve are noted corresponding values of A_1/a. Thus for A_1/a equal to infinity, we have $1/J_{NEIII_0} = 1$.

Let us examine the nonlinear characteristic shown in Figure 11. Such a characteristic naturally belongs to any servo-amplifier of limited power in the absence of an insensitivity zone. For absolute x_1 much smaller than a we have $x_2 = Nx_1$ where $N = B/a$. For absolute x_1 much greater than a, we have absolute $x_2 = B$ = const.

In accordance with (4) and (5) we get

$$\begin{aligned} J_{NEIV} &= g_{ek}(A_1/a) = \frac{4}{\pi A_1}\int_0^{\pi/2} \Phi(A_1 \sin\psi)\cdot \sin\psi \, d\psi \\ &= N\cdot\frac{4}{\pi}\left[\frac{\alpha}{2} - \frac{1}{4}\sin 2\alpha + \frac{\cos\alpha}{(A_1/a)}\right] \\ &= N\cdot J_{NEIV_0} \end{aligned} \tag{13}$$

where

$$J_{NEIV0} = \frac{4}{\pi}\left[\frac{2}{\alpha} - \frac{\sin 2\alpha}{4} + \frac{\cos\alpha}{(A_1/a)}\right]$$

and

$$\alpha = \text{arc} \sin(a/A_1)$$

Figure 12 is a graph of equivalent admittance J_{NEIV0} versus the ratio A_1/a. Its maximum value is equal to 1 for A_1/a much smaller than 1 and following this it monotonically decreases to zero with increase in A_1/a.

Let us examine the nonlinear characteristic shown in Figure 13a. It differs from the characteristic in Figure 11 by the presence of an insensitivity zone. For abs. x_1 less then a, we have $x_2 = 0$; for x_1 between a and na, we have $x_2 = (B/a(n-1)) \cdot (x_1 - a)$; for x_1 greater than na, we have $x_2 = B$; for x_1 between -na und -a, we have $x_2 = (B/a(n-1)) \cdot (x_1 + a)$; for x_1 much less then -na, we have $x_2 = aB$.

In accordance with (4) and (5) we get:

$$\begin{aligned} J_{NEV} &= g_{ek}(A_1/a, n) = \frac{4}{\pi A_1}\int_0^{\pi/2}\phi(A_1 \sin\psi)\cdot\sin\psi\, d\psi \\ &= \frac{4B}{\pi a(n-1)}\left[\frac{\alpha_2-\alpha_1}{2} - \frac{\sin 2\alpha_2 - \sin 2\alpha_1}{4} - \frac{\cos\alpha_1 - \cos\alpha_2}{(A_1/a)} + \frac{\cos\alpha_2}{A_1/a(n-1)}\right] \\ &= N \cdot J_{NEV_0} \end{aligned} \tag{14}$$

where $N = B/a(n-1)$ is the curvature of the characteristic in the rectilinear part,

$$J_{NEV_0}(A_1/a, n) = \frac{4}{\pi}\left[\frac{\alpha_2-\alpha_1}{2} - \frac{\sin 2\alpha_2 - \sin 2\alpha_1}{4} - \frac{\cos\alpha_1 - \cos\alpha_2}{(A_1/a)} + \frac{\cos\alpha_2}{A_1/a(n-1)}\right]$$

$$\alpha_1 = \arcsin(a/A_1) \text{ and } \alpha_2 = \arcsin(na/A_1)$$

Figure 14 is a graph of equivalent admittence J_{NEVO} versus the ratio A_1/a for various values of n.

For A_1 very much smaller than na, the values of J_{NEVO} naturally coincide with the values of J_{NEIO} (see Figure 3). For A_1 very much larger than na, the values of J_{NEVO} approach the values of J_{NEIVO}.

Let us examine the nonlinear characteristic shown in Figure 15a. It differs from the characteristic in Figure 4 by the presence of the hysteresis effect. During variation of x_1 within the interval 0 to a, x_2 remains equal to 0. For x_1 greater than a, x_2 is constant and equal to B. With decrease in x_1 to the value b = ma, x_2 remains constant and equal to B. For x_1 = ma, x_2 abruptly approaches 0 by a jump. The quantity x_2 is an odd function of x_1. Figure 15 shows an elementary scheme having a similar characteristic. The polarized relay RP has a slide contact k_1, which depending upon the influence and direction of the current in the winding of the relay RP either is connected with the immobile contact k_2' or k_2'' or is located in a neutral position between them. During variation in position of brush F, the voltage in the winding of the relay RP, determined by coordinate x_1, varies. The relay operates for some certain value of the voltage; that is, contact k_1 connects with either contact k_2' or k_2''. Between the points c and d the voltage x_2 is switched on. By virtue of the probable presence, in relay RP, of a coefficient of recovery different from one the contacts will break for a voltage in the relay smaller than the working voltage. In this way, the characteristic of the studied system -- that is, the dependence of voltage x_2 (in relative terms) upon the voltage x_1 (in relative terms) -- will have the form shown in Figure 15a. The double-valued dependence between x_1 and x_2 predetermines the presence of active and reactive conductance in the equivalent admittance of the studied nonlinear element. In accordance with equations (5a) and (5b) we obtain

$$\begin{aligned} g_{ek}(A_1/a, m) &= \frac{2}{\pi A_1}\int_0^{\pi}\phi(A_1 \sin\psi)\cdot\sin\psi\, d\psi \\ &= \frac{2B}{\pi a(A_1/a)}\cdot(\cos\alpha_1 - \cos\alpha_2) \\ b_{ek}(A_1/a, m) &= \frac{2}{\pi A_1}\int_0^{\pi}\phi(A_1 \sin\psi)\cdot\cos\psi\, d\psi \\ &= \frac{2B}{\pi a(A_1/a)}\cdot(\sin\alpha_2 - \sin\alpha_1) \end{aligned}$$

where $\alpha_1 = \arcsin(a/A_1)$ and $\alpha_2 = \pi - \arcsin(ma/A_1)$

hence

$$
\begin{aligned}
J_{NEVI}(A_1/a, m) &= g_{ek} + j\,b_{ek} \\
&= \sqrt{g_{ek}^2 + b_{ek}^2} \cdot \exp(j\gamma) \\
&= \frac{4B}{\pi a (A_1/a)} \cdot \sin\frac{\alpha_2 - \alpha_1}{2} \cdot \exp(j\gamma) \\
&= N\,J_{NEVI_0}(A_1/a, m)
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
N &= B/a \\
J_{NEVI_0}(A_1/a, m) &= \frac{4}{\pi} \cdot \frac{1}{(A_1/a)} \cdot \sin\frac{\alpha_2 - \alpha_1}{2} \cdot \exp(j\gamma) \\
&= \left|J_{NEVI_0}\right| \cdot \exp(j\gamma) \\
\left|J_{NEVI_0}\right| &= \frac{4}{\pi} \cdot \frac{1}{(A_1/a)} \cdot \sin\frac{\alpha_2 - \alpha_1}{2} \\
&= \arctan(b_{ek}/g_{ek}) \\
&= \arctan\frac{\sin\alpha_2 - \sin\alpha_1}{\cos\alpha_1 - \cos\alpha_2} = \frac{\pi}{2} - \frac{\alpha_2 + \alpha_1}{2}
\end{aligned}
$$

Figure 16 gives graphs of J_{NEVIO} and gamma versus the ratio A_1/a for various values of m.

Let us determine the reciprocal of $J_{NEVIO}(A_1/a, m)$; namely, the equivalent impedance:

$$
\begin{aligned}
1/J_{NEVIO} &= (1/N) \cdot \frac{1}{g_{ek} + j\,b_{ek}} \\
&= (1/N) \cdot \left[g_{ek}/(g_{ek}^2 + b_{ek}^2) - j\,b_{ek}/(g_{ek}^2 + b_{ek}^2)\right] = X + jY
\end{aligned}
\tag{16}
$$

The curve traced by the terminal point of the equivalent-impedance vector for a number of values of m is given in Figure 17. The values of X are plotted on the abscissa axis and jY on the ordinate axis.

The Equations of a Regulation System with Nonlinear Elements

If in a regulation system there is one nonlinear element, then it can always be represented as a combination of linear and nonlinear elements (Figure 18); hence the linear element, in its characteristic, unites the entire linear part of the regulation system.

The first approximation equations of a regulation system can be written in the following form:

$$x_2 = J_o(p)x_1 \tag{17}$$

$$x_3 = J_{NE}(A_1/a)x_2 \tag{18}$$

$$x_3 = x_1 \tag{19}$$

Hence the system's equations of free oscillations will be:

$$J_c(p) \cdot J_{NE}(A_1/a) - 1 = 0 \tag{20}$$

To find the parameters of possible stationary (steady-state) oscillations, we substitute $p = j\omega$ in equation (20) [4, 6]. (Note: ω is the cyclic frequency or angular velocity.)

Thus we get:

$$J_o(j\omega) \cdot J_{NE}(A_1/a) = 1 \qquad (21)$$

Setting separately the real and imaginary parts of equation (21) to zero, we get two equations with two unknowns: the cyclic frequency (angular velocity) ω and amplitude A_1/a (in relative units) of auto-oscillation. If as a result of solving these equations we obtain real values for ω, then auto-oscillations in the system are possible. The question of stability of auto-oscillations is separately studied below.

The same solution can be conducted graphically by using the frequency characteristic of the linear part of the regulation system. Let us rewrite equation (21) in the following form:

$$J_o(j\omega) = 1/J_{NE}(A_1/a) \qquad (21')$$

The left part of this equation, with ω varying between negative and positive infinity, represents the phase-amplitude characteristic of the regulation system's linear part in a disconnected (open) state and will be employed in solving by Nyquist's method the problem of the system's stability in a closed state. The right part of the equation represents the equivalent impedance of a nonlinear element whose characteristic can generally be represented, for A_1/a between zero and infinity, in the form of a curve with the same coordinates as the amplitude-phase characteristic. The intersection of the phase-amplitude characteristic with the characteristic of the equivalent impedance determines the frequency and amplitude of possible oscillations. If the characteristics do not intersect, then it means that there is no real value of the cyclic frequency ω satisfying equation (21); consequently the studied regulation system, in the first approximation, cannot have steady-state oscillations of finite amplitude (distinct from zero or infinity), as determined by the nonlinear element.

Stability of Auto-Oscillations

Let the amplitude-phase characteristic of a stable linear system and the characteristic of an equivalent impedance intersect at more than one point. In Figure 19 these characteristics intersect at two points, N and M. The point N corresponds (according to the characteristic of $J(j\omega)$) to the frequency $\omega = \omega_1$ and (according to the characteristic of $1/J_{NE}$) the amplitude of auto-oscillation A_1/a; to the point M correspond $\omega = \omega_2$ and $A_1/a = (A_1/a)_2$.

We will show that, for a given form of the characteristics, to the point N correspond unstable auto-oscillations, while to point M there correspond stable ones.

In a given case, it is sufficient to examine the stability of oscillations with respect to the amplitude.

Let us assume that the amplitude of auto-oscillations corresponding to point N has increased slightly and become equal to $(A_1/a)_1 + d(A_1/a)$. Then the equivalent impedance will be determined by the point N_1 and will by equal to $(1/J_{NE})II$. The common amplification factor of a regulation system in a closed state is defined as the product of the amplification factors of the linear and the nonlinear elements. For the amplitude $(A_1/a)_1 + d(A_1/a)$ of oscillation, the common amplification factor will be equal to

$$K_{NI} = J(j\omega) \cdot J_{NE1I}$$

The amplitude-phase characteristic corresponding to this amplification factor intersects the real axis at $\omega = \omega_{11}$ (frequency corresponding to the point of intersection of the ray ON_1 with the characteristic of $J(j\omega)$) to the right of point 1, j0 or, as seen from Figure 19, we have:

$$J(j\omega_{11}) \text{ is greater than } 1/J_{NE11}$$

In this way the amplitude-phase characteristic of $K_{N1}(j\omega)$ envelop point 1, j0 and consequently the system turns out to be unstable; that is, the oscillation will increase. The process characterized by point N will be unstable.

If we assume that the amplitude of oscillations decreases, then the equivalent impedance will be, let us say, determined by point N_2 and will equal $1/J_{NE12}$.

A system's common amplification factor for decreased amplitude will be equal to

$$K_{N2} = J(j\omega) \cdot J_{NE12}$$

The amplitude-phase characteristic corresponding to this amplification factor intersects the real axis when $\omega = \omega_{12}$ (see Figure 19) to the left of the point (1, j0) or, as seen from Figure 19, we have

$$J(j\omega_{12}) \text{ less than } 1/J_{NE12}$$

In this way the characteristic of $K_{N2}(j\omega)$ does not envelop point 1, j0; that is, the system is stable and hence the oscillations will become damped.

Hence it follows that if the initial deviations in the system are such that the deviations at input of the nonlinear element are less than $(A_1/a)_1$, then oscillations will not develop in the system.

Similar reasoning can show that oscillations corresponding to point M are stable. Actually, let us assume that the amplitude of oscillations has increased somewhat and hence become equal to $(A_1/a)_2 + d(A_1/a)$. Then the equivalent impedance of the nonlinear element will be determined by point M_1 and will be equal to $(1/J_{NE})_{21}$.

The system's common amplification factor for an increased amplitude will equal $K_{M1} = J(j\omega) \cdot J_{NE21}$. The amplitude-phase characteristic of $K_{M1}(j\omega)$ will intersect the real axis at $\omega = \omega_{21}$ to the left of point 1, j0 or, as can be seen from Figure 19, we have

$$J(j\omega_{21}) \text{ less than } 1/J_{NE21}$$

and consequently the system becomes stable in the sense that the oscillations start to decrease. If we assume that the amplitude of oscillations has decreased, then it is easy analogously to show that conditions are created that cause oscillations to begin to increase and the process slips back into the scheme characterized by point M.

In the general case, if the amplitude-phase characteristic of the system's linear part has an arbitrary form and intersects the impedance characteristic of the nonlinear element, then the problem of stability of oscillations with frequency ω and reduced (dimensionless) amplitude A_1/a corresponding to the point of intersection of the characteristics is solved by plotting the characteristic of the system's general amplification factor in the open (disconnected) stage against the increased amplitude $J(j\omega) \cdot J_{NE}\,[(A_1/a) + d(A_1/a)]$. If this characteristic encompasses point 1, j0, then the oscillations are unstable; if not, they are stable. (Note: It is assumed that the linear system in a close state is stable.)

Concrete Examples of a Regulation System With Nonlinear Elements

1. The Case Where an Element With Coulomb (dry) Friction is Present

Let us examine a regulation system in which the nonlinear element with Coulomb friction is connected successively with the linear part of the regulation system, in accordance with the system in Figure 18. (Note: Such a separation of a nonlinear element with the characteristic shown in Figure 6a is admissible if the movement in this element, in the case of the absence of friction, is defined by a zero-order equation.) The analysis of such a system (in the first approximation) is reduced to the examination of equation (21'):

$$J_o(j\omega) = 1/J_{NEIII} \qquad (21')$$

More descriptive results, as were already shown, can be obtained graphically. It is necessary, con equently, to examine the possible common (simultaneous) dispositions of the amplitude-phase characteristic of $J_o(j\omega)$ and the characteristic of the equivalent impedance $1/J_{NEIII}(A_1/a)$ (that is, to find the intersections of their curves). The characteristic of an equivalent impedance of a nonlinear element with Coulomb friction (for $N = 1$) is strictly determinate; it is given by equation (12) and is shown in Figure 10.

Therefore, the common (simultaneous) disposition of the characteristic is fully determined by the form of the amplitude-phase characteristic of $J_o(j\omega)$.

Several cases are possible; some of them are shown in Figure 20. In Figure 20a is shown the amplitude-phase characteristic of the linear part of the regulation system $J_o(j\omega)$ which does not compass the point 1, j0 and does not intersect the characteristic of the equivalent impedance $1/J_{NEIIIO}$. Such a system, being stable in the absence of a nonlinear element, remains stable even with its presence. Similar systems we will call absolutely stable.

In Figure 20b is shown the characteristic of $J_o(j\omega)$, which does not encompass the point 1, j0, but intersects the characteristic of the equivalent impedance $1/J_{NEIIIO}$ at point M. This point determines the frequency and amplitude of the oscillations that arise. These oscillations are stable, in accordance with conditions examined in the section "Stability of Auto-Oscillations."

Thus the studied system that is stable in the absence of a nonlinear element turns out to be unstable (in the sense of developing oscillations) in the presence of a nonlinear element. Such a system will be called unstable.

In Figure 20 is shown the characteristic of $J_o(j\omega)$, which intersects the characteristic of the equivalent impedance at point N and encompasses the point 1, j0. The point N corresponds to unstable oscillations. This means that if the initial deviations are smaller in magnitude than those corresponding to point N, then oscillations do not arise in the system; if, however, they are greater, then the amplitude of oscillations will increase and cannot be limited by the given nonlinear element. Such a system we will call conditionally stable.

In Figure 20d is shown a case where the characteristic intersect at two points: N and M. The first of these two corresponds to unstable oscillations and the second, to stable. Thus if the initial deviations are smaller than those corresponding to point N, then oscillations in the system do not arise; if they are greater, oscillations arise, depending upon the parameters of point M. Such a system we shall call conditionally stable. As long as the characteristic of $J_o(j\omega)$ does not encompass point 1, j0 in a given case, the studied system is stable in the absence of a nonlinear element.

In Figure 20e is shown the case where characteristics do not intersect; here the characteristic of $J_o(j\omega)$ encompasses the characteristic of $1/J_{NEIII}$ on the right. In this case a linear system that is unstable during the absence of a nonlinear element also remains unstable during its presence. Such a system we shall call absolutely unstable.

a. Primary Regulation of an Inertia Object by a Regulator Having Friction in the Measuring Element [1]

The schematic of a regulated system is shown in Figure 21. The equation of the object of regulation has the form:

$$\phi = \frac{1}{k + pT_a}\xi \tag{22}$$

The regulator is assumed not to have any mass or viscous friction. Its measuring element has Coulomb friction and acts simultaneously as an amplifier. The equation of the regulator's measuring element has the form:

$$\eta = \frac{1}{\delta}\phi \tag{23}$$

In the schematic the nonlinear element NEIII is isolated separately; the nonlinear element shows the presence of dry friction in the regulator. The relation between η_H and η is given by the characteristic shown in Figure 21b. The equation of this characteristic can be presented in the following form (see "1" under section "Equivalent Admittances"):

$$\eta_H - \eta = \begin{cases} +a & \text{for } \dot{\eta}_H < 0 \\ -a & \text{for } \dot{\eta}_H < 0 \end{cases}$$

$$|\eta_H - \eta| < a \quad \text{for } \dot{\eta}_H = 0 \tag{24}$$

The equation of the servomotor has the form:

$$\xi = -\frac{1}{pT_a}x_1 \tag{25}$$

The general amplification factor of the linear part of the regulation system will be equal to:

$$J_0(p) = \frac{\eta}{x_1} = -\frac{1}{pT_c\delta}\cdot\frac{1}{k + pT_a} \tag{26}$$

Let us transform equation (26), for which we will introduce a new operator:

$$q = p\,T_a/k \tag{27}$$

Then equation (26) may be written in the form:

$$J_0(p) = \frac{T_a}{T_c\delta k^2}\cdot\frac{(-1)}{q(1+q)} = NJ_0(q) \tag{26'}$$

where

$$N = \frac{T_n}{T_c\delta k^2} \text{ and } J_{01}(q) = -\frac{1}{q(1+q)}$$

The amplitude-phase characteristic of the linear part of the regulation system will be obtained by substituting the expression $j\Omega$ for q, where Ω is the reduced (dimensionless) cyclic frequency.

$$J_0(j\Omega) = N[-1/j\Omega(1+j\Omega)] = NJ_{01}(j\Omega) \tag{28}$$

Figure 22 shows the amplitude-phase characteristic of $J_{01}(j\Omega)$ and the characteristic of the equivalent impedan. NEIII. The common (simultaneous) disposition of the characteristics are such that in the given case we have to deal with an absolutely stable system. For N = 3 approximately, the characteristics touch each other. This value of N is a lower limit. For N greater than 3 the characteristics intersect; in the system oscillations arise whose amplitude and frequency are determined by the point of intersection of the characteristics. Thus the possibility of oscillations developing in the system is determined by the magnitude of N and does not depend upon the magnitude of friction, determined by coordinate a. For N greater than 3, oscillations develop in the system; hence their cyclic frequency $\omega = \Omega \cdot (k/T_a)$ (see equation (27)) and amplitude A_1 will depend upon both the magnitude of N and the magnitude of a, since the point of intersection of the characteristics determines the values of Ω and A_1/a.

In Figure 23 are presented curves of Ω and A_1/a versus the magnitude of N, obtained by graphically solving the problem. The larger N, the larger Ω and A_1/a, as it directly follows from Figure 22.

b. Primary Regulation by a Regulator Having Mass and Viscous and Dry Frictions (Vyshnegradskiy's Problem) /7/

A schematic of a regulated system is shown in Figure 24. The equation of the object has the form:

$$\eta = + \frac{1}{pT_0}\xi \tag{29}$$

The equation of the regulator (without consideration of dry friction) is

$$\xi = -\frac{1}{\frac{T^2}{4}\cdot p^2 + \frac{T_1}{2}\cdot p + \delta} X. \tag{30}$$

In the schematic the nonlinear element NEIII is isolated separately; the element shows the presence of dry friction in the regulator. The relation between η_H and η is determined by the characteristic shown in Figure 21b and equation (24), if we limit ourselves to studying the interrupted movement of the regulator's clutch (the regulator's clutch comes to a stop at the end of each swing because of friction).

The general amplification factor of the linear part of the regulation system will be equal to:

$$J_0(p) = \frac{\eta}{X_1} = -\frac{1}{pT_0\left(\frac{T^2}{4}p^2 + \frac{T_1}{2}p + \delta\right)} \tag{31}$$

Let us transform equation (31), for which we will introduce the new auxiliary operator:

$$q = p \cdot \sqrt[3]{\frac{T_0 T^2}{4}} \tag{32}$$

Then the equation (31) takes the form

$$J_0(q) = -1/q(q^2 + Bq + A) \tag{31'}$$

where

$$A = (\delta) \sqrt[3]{\frac{4T_0^2}{T^2}}$$

and

$$B = (T_1/T) \cdot \sqrt[3]{\frac{2T_0}{T}}$$

The amplitude-phase characteristic of the linear part of the regulation system, as follows from equation (31'), will be

$$J_0(j\Omega) = -\frac{1}{j\Omega(-\Omega^2 + A + jB\Omega)} \tag{33}$$

That is to say, it is wholly determined by two generalized parameters, A and B.

From equation (33), in accordance with Nyquist's criterion, it is easy to discern that the linear part of the system will be stable if A, B, and (AB-1) are greater then zero. (34) The first two conditions will always be fulfilled; the third determines the limiting relation between the parameters of the regulator and object. Substituting in the last condition the values of A and B, we get:

$$\frac{2\delta T_0 T_1}{T^2} > 1 \tag{34'}$$

From the conditions of stability (34) it follows that the amplitude-phase characteristic of the linear part of the regulation system $J_0(j\Omega)$ for AB greater than 1 will not encompass the point 1, j0; for AB = 1, it will pass through it; and for AB less than 1, it will encompass it.

In Figure 25a is shown the common (simultaneous) disposition of the amplitude-phase characteristic of the system's linear part for AB greater than 1 and the characteristic of the equivalent impedance of the nonlinear element NEIII.

The characteristics do not intersect; hence the characteristic of $1/J_{NEIII0}$ lies wholly right of the characteristic of $J_0(j\Omega)$. Consequently, the system in this case is absolutely stable. For AB less than 1 the characteristics are located in accordance with Figure 25b: one point of intersection exists; namely, point N corresponding to unstable oscillations.

If, consequently, the initial deviation is smaller than those corresponding to point N, oscillations do not arise in the system; that is, the process converges. If greater, oscillations arise; that is, the process diverges. Therefore the amplitude of the increasing oscillations cannot be limited by type-NEIII nonlinearity just studied. For AB much less than 1, the characteristics are formed in accordance with Figure 25c; that is, they do not intersect and therefore the characteristic of $1/J_{NEIII0}$ lies wholly to the left of $J_0(j\Omega)$; that is, the system is absolutely unstable.

The examination of the presented examples allows us to make, in particular, the following conclusions. Second order systems which remain stable without nonlinear elements for any amplification factor become unstable during the presence of type-NEIII nonlinearity (which creates additional phase-shift) for a finite amplification factor. Systems above the second order of type-NEIII nonlinearity also change essentially the conditions of stability.

2. The Case Where a Nonlinear Element With a Zone of Insensitivity Is Present

Nonlinear elements with an insensitivity zone are often found in regulation systems. As a practical example of such a nonlinear element, it is possible to point to the measuring element of a regulator whose sliding contact is located in a fork and which, during a variation in the regulated quantity greater (variation) than the zone of insensitivity, switches on one or the other side of a constant-speed servomotor.

Let us examine a regulation system made up of a linear element and a nonlinear element of type NEII (Figure 26). Let the linear part of the regulation system be characterized by the operator admittance $J_0(p)$ and let the nonlinear part be represented by the equivalent admittance J_{NEII} (see "2" under section "Equivalent Admittances"). The equation of the regulation system will have the form:

$$x_2 = J_{NEII} x_1$$

$$x_3 = J_o(p)\ x_2$$

$$x_3 = x_1 \qquad (35)$$

The equation of the system's free oscillations will be:

$$J_o(p) \cdot J_{NEII} = 1 \qquad (36)$$

In the regulation system nondamped oscillations can arise if equation (36) is true for a purely imaginary value of p; that is if

$$J_o(j\omega) = 1/J_{NE} \qquad (37)$$

where ω is a real number. Replacing J_{NEII}, in accordance with equation (7), by NJ_{NEIIO}, we get

$$N \cdot J_o(j\omega) = 1/J_{NEIIO} \qquad (37')$$

Let a regulation system without a nonlinear element be characterized by the amplitude-phase characteristic of $J_o(j\omega)$ (see Figure 27). Let us plot in Figure 27 the vector of the equivalent impedance $1/J_{NEIIO}$ for various values of A_1/a. Since its phase equals zero, the termini of this vector will lie on the real axis to the right of the point (1/0.637, j0). We will denote by K the modulus of $J_o(j\omega_b)$, where ω_b is the value of the "intersection frequency," at which the amplitude-phase characteristic intersects the real axis.

The condition necessary for the system to be stable will be the nonintersection of the characteristic of $NJ_o(j\omega)$ with the straight line $1/J_{NEIIO}$ which, in accordance with (37), will hold for

$$NK \text{ less than } 1/0.637 \qquad (38)$$

or

$$K \text{ less than } (1.57)/N$$

If K is greater than 1.57/N, then oscillations develop with a frequency ω_o and an amplitude whose magnitude is determined by the point of intersection of the characteristic of $NJ_o(j\omega)$ with the straight line $1/J_{NEIIO}$.

As follows from the section "The Concept of Equivalent Admittance of a Nonlinear Element" and from the form of the characteristic of the equivalent admittance of the nonlinear element J_{NEIIO} (see Figure 5), stable oscillations are possible only for the relative amplitude A_1/a greater than 1.41. The graph of the ratio A_1/a versus KN is shown in Figure 28.

As a practical example, let us examine the operation of a SN91-type voltage regulator [8].

The schematic of a regulated system is shown in Figure 29a. The regulated object consists of an exciter 1 and generator 2. The regulator consists of a measuring element 3 with a zone of insensitivity and a servomotor 4. First let us study the behavior of the system in the case where the servomotor has a constant speed. The characteristic of the nonlinear element in the schematic will in this case take the form shown in Figure 29b.

Let us make the simplifying assumptions that the velocity of the servomotor instantaneously attains its final value when the contacts of the measuring element are closed. Calculating the inertia of the servomotor would somewhat complicate the expression for the amplification-factor operator of the linear parts of the regulation system; that is, it would be necessary to consider the magnitude of e_5 as proportional to the voltage at the rotor of the servomotor and not to the velocity.

Let us write the equations for the system:

Equation for the exciter $e_2 = b_1 e_1/(1+pT_e)$ (1)

Equation for the generator $e_3 = b_2 e_2/(1+pT_g)$ (2)

Equation for the measuring element allowing for the damper $e_4 = b_3 e_3/(1+pT_d)$ (3)

Equation for the nonlinear element $e_5 = J_{NEII} \cdot e_4 = NJ_{NEIIO} \cdot e_4$ (4)

Equation for the servomotor $e_1 = -(1/pT_s)x$ (5)

Equation for feedback $x = e_5$ (6)

(1)–(6): (39)

where b_1, b_2, and b_3 are the amplification factors of the exciter, generator, and measuring element respectively. The time constants are T_e, T_g, T_d, T_s for the exciter, generator, damper, servomotor, respectively. $N = 1/a$ where a is the relative variation in the regulated voltage during which the measuring element's contacts are closed.

The amplification factor of the regulation system's linear part in the open state will equal:

$$J_o(p) = e_4/x = -b_1b_2b_3/pT_s\,(1+pT_e)(1+pT_g)(1+pT_d) \tag{40}$$

The system is astatic and has three inertia members.

The length of the segment cut off by the amplitude-phase characteristic of the linear system on the real positive axis is given by the equation:

$$K = \mu_s \frac{1 + {}_3S_1 - {}_3S_2 \cdot \Omega_o^2}{(1+\Omega_o^2)(1+\Omega_o^2 \cdot \tau_2^2)(1+\Omega_o^2 \cdot \tau_3^2)} \tag{41}$$

where

$$\mu_s = \frac{b_1b_2b_3}{\tau_s};\ \tau_s = \frac{T_s}{T_g};\ \tau_2 = \frac{T_v}{T_g};\ \tau_3 = \frac{T_d}{T_g}$$

$${}_3S_1 = \tau_2 + \tau_3;\ {}_3S_2 = \tau_2\tau_3;\ \Omega_o^2 = \frac{1}{{}_3S_{12}};\ \Omega_o = \omega_o T_g;\ {}_3S_{12} = \tau_2 + \tau_3 + \tau_2\tau_3$$

According to (38) the regulation system (taking the nonlinear element into account) will be stable for K less than 1.57/N or 1.57 a.

It follows from the latter inequality that to have stability it is necessary that

$$aT_c > 0.637\, b_1b_2b_3 T_g \frac{1 + {}_3S_1 - {}_3S_2\Omega_o^2}{(1+\Omega_o^2)(1+\Omega_o^2\tau_2^2)(1+\Omega_o^2\tau_3^2)}$$

In general it can be assumed that $b_1 = b_2 = b_3 = 1$.
Now let $T_g = 2$ sec; $T_e = 0.4$ sec; $T_d = 0.1$ sec.
Then $\Omega_o^2 = 3.85$ and $\omega_o = \omega_o/T_g = 0.98\ \text{sec}^{-1}$

$$\Omega_o^2 = 3.85,\ \omega_o = \frac{\Omega_o}{T_g} = 0.98\ \text{sec}^{-1},$$

and for stability it is necessary that

$$aT_s \text{ be greater than } 0.637 \times 0.432 = 0.275 \text{ sec.}$$

If there is no damper ($T_d = \tau_3 = 0$), then:

$$\Omega_0^2 = 5 \text{ and } \omega_0 = \Omega / T_g = 1.12/\text{sec}$$

and for stability it is necessary that:

$$aT_s \text{ be greater than } (0.637) \cdot (0.432) = 0.275 \text{ sec}\cdot$$

If it be assumed that the zone of insensitivity amounts to 2 percent ($a = 0.02$), then the time T_s of the servomotor must be greater than 13.75 sec when there is a damper and greater than 10.6 sec if there is no damper.

Let us examine the behavior of a voltage-regulation system when there is a zone of insensitivity and a servomotor of variable velocity. In this case the characteristic of the schematic's nonlinear element can be represented by the graph in Figure 29c.

According to the characteristics of the setting of the SN-01 regulator, we have $a = 0.01$ and $na = 0.04$. The equation of the nonlinear element will take the form:

$$e_5 = J_{NEV} \cdot e_4 = NJ_{NEVO} \cdot e_4$$

$$N = 1/a(n-1) = 33.3$$

The condition necessary for stability, according to the inequality (41), will be

$$K \text{ less than } 1/N(J_{NEVO})_{max}$$

The maximum value of (J_{NEVO}) when $n = 4$ is 0.68 (see Figure 14).

From equation (41) and inequality (38) we obtain:

$$T_s > (0.68)\cdot(33.3)\, b_1 b_2 b_3 T_g \cdot \frac{1 + {}_3S_1 - {}_3S_2\,\Omega_0^2}{(1+\Omega^2)(1+\Omega_0^2\tau_2^2)(1+\Omega_3^2\tau_3^2)} \qquad (42)$$

For the values assumed above, we find from (42) that for stability it is necessary that T_s be greater than (0.68) (33.3) (0.432) = 9.8 sec when there is a damper and that T_s be greater than (0.68) (33.3) (0.33) = 7.6 sec when there is no damper ($T_d = 0$).

In practice, T_s is located in the interval 8 - 10 sec, which corresponds to the 12 - 15 sec of time of the full course of the controller's traverse (from rest position to rest position). The full course of the traverse is calculated on a variation of about 150 percent in the exciter.

By comparing the results of calculations for a constant-speed servomotor and for a variable-speed servomotor, we see the advantages of the following: stability is attained for small values of T_s during: (a) simultaneous decrease in the zone of insensitivity (instead of $a = 0.02$ in the first case, we have $a = 0.01$ in the second) and, consequently, (b) increase in the accuracy of regulation.

3. The Case Where Two Nonlinear Elements are Series-Connected

Regulation systems are often found where not one but two or more nonlinear elements are connected in series with a linear part. Let us demonstrate the method of examination proposed below by an example of two nonlinear elements connected in series, one of which is an element with free play or friction (NEIII), while the other is characterized by a zone of insensitivity and a limited proportional dependence (NEV).

Figure 30a shows a schematic of two nonlinear elements connected in series. Figure 30b shows the characteristic of these elements.

The equations connecting x_2 and x_1 can be written as follows (see equation 9):

$$x_2 = N_1(x_1 - x_{10}) \text{ for } \dot{x}_2 \text{ greater than } 0,$$

$$x_2 = N_1(x_1 + x_{10}) \text{ for } \dot{x}_2 \text{ less than } 0,$$

$$\text{abs.}(x_2 - N_1x_1) \text{ less than } N_1x_{10} \text{ for } \dot{x}_2 = 0$$

The equations connecting x_3 and x_2 have the following form:

$$x_3(x_2) = -x_3(-x_2) = \begin{cases} 0 & \text{for abs. } x_2 \text{ less than } x_{21}, \\ N_2(x_2 - x_{21}) & \text{for } x_{21} \text{ less than } x_2 \text{ less than } x_{22} \\ B & \text{for } x_2 \text{ greater than } x_{22}. \end{cases}$$

In accordance with the given characteristics, it is possible to plot the characteristics of $x_3(x_1)$ which will correspond to the characteristics of the reduced nonlinear element equivalent to the first two.

Figures 30c and 30d introduce the characteristics of $x_3(x_1)$ for the two values of $x_{1\,max}$; namely, the amplitude of variation of x_1. In Figure 30c, $x_{1\,max}$ is less than x_{12}; in Figure 30d, $x_{1\,max}$ is greater than x_{12}.

The equations connecting x_3 and x_1 take the following form:
1) For x_{11} less than x_{1max} less than x_{12} (Figure 30c)

	Sect
$x_3 = N_1N_2(x_1 - x_{11})$	1-2
$x_3 = N_1N_2(x_{1max} - x_{11})$	2-3
$x_3 = N_1N_2(x_1 - x_{11} + 2x_{10})$	3-4
$x_3 = 0$	4-5
$x_3 = N_1N_2(x_1 + x_{11})$	5-6
$x_3 = N_1N_2(-x_{max} + x_{11})$	6-7
$x_3 = N_1N_2(x_1 + x_{11} - 2x_{10})$	7-8
$x_3 = 0$	8-1

2) For x_{1max} greater than x_{12} (Figure 30d)

$x_3 = N_1N_2(x_1 - x_{11})$	1-2
$x_3 = B$	2-3
$x_3 = B$	3-4
$x_3 = N_1N_2(x_1 - x_{11} + 2x_{10})$	4-5
$x_3 = 0$	5-6
$x_3 = N_1N_2(x_1 + x_{11})$	6-7

$x_3 = -B$ 7-8

$x_3 = -B$ 8-9

$x_3 = N_1N_2(x_1 + x_{11} - 2x_{10})$ 9-10

$x_3 = 0$ 10-1

where $x_{11} = x_{10} + x_{21}/N_1$ and $x_{12} = x_{1} + x_{22}/N_1$.

Let us introduce the notation: $n = x_{11}/x_{10}$ and $m = x_{12}/x_{11}$ and, in accordance with equations (4) and (5), let us determine the equivalent admittance $J_{NEPr} = N_1N_2J_{NERe0}$ of the reduced nonlinear element fixed by the characteristics in Figures 30c and 30d. The results of calculation for various values of n for m = 2 are shown in Figure 31, where the modulus and phase of J_{NERe0} are given as functions of $A_1/a = x_{1max}/n \cdot x_{10}$. Figure 32 gives the vector loci for the equivalent impedance $1/J_{NERe0}$, of the reduced nonlinear element, for various values of n when m = 2. (Note: Re stands for "reduced.") Without a zone of insensitivity ($x_{21} = 0$; n = 1), if the amplitude of variation of x_1 is less than x_{12}, the characteristic of the reduced nonlinear element will coincide with the characteristic of the nonlinear element NEIII. At large amplitudes this characteristic will be modified. This modification of characteristics can also readily be seen from the form of the characteristics of the impedance of the reduced nonlinear element shown in Figure 32.

Let there be a regulation system composed of: (a) a linear element that is characterized by the operator admittance $J_o(p)$, and (b) elements of the NEIII and NEV type (Figure 33a). In accordance with the foregoing, the given system can be replaced by an equivalent (Figure 33b). The behavior of the system will be determined by the relative disposition of the amplitude-phase characteristic of $N_1N_2J_o(j\omega)$ and the characteristic of the impedance $1/J_{NERe0}$; the disposition will depend on N_1, N_2, n and m, for a given $J_o(j\omega)$.

As an example, let us examine a system containing an automatic variable-pitch screw /9/. The equations of this system will be:

Equation of the motor $d\omega/dt = -M\omega - N\phi$

Equation of the indicator (neglecting inertia) $c\eta - b\omega = \theta(\dot{\eta}) = \begin{cases} -k & \text{for } \dot{\eta} > 0 \\ +k & \text{for } \dot{\eta} < 0 \\ & \text{for } \dot{\eta} = 0 \end{cases}$ $|c\eta - b\omega| < k$

Equation of the servo-motor $d\phi/dt = F(\eta) = -F(-\eta) = \begin{cases} 0 & \text{for } |\eta| < \eta_0 \\ \alpha(\eta - \eta_0) & \text{for } \eta_0 < \eta < \eta_1 \\ A & \text{for } \eta > \eta_1 \end{cases}$

Schematic of a system corresponding to the given equations is shown in Figure 34. The operator admittance of the linear part will equal:

$$J_o(p) = J_1(p) \cdot J_2(p) = -(N/(p+M)) \cdot (1/p) = -N/Mp(1+p/M)$$

Making the substitution $q = p/M$, we obtain:

$$J_o(q) = -N/M^2q(1+q)$$

If there were no type-NEV nonlinearity ($J_{NEV0} = 1$) then the system under examination would be stable for the condition $(N/M^2) \cdot (b/c) \cdot$ (alpha) less than 3 (see equation 26'). The values of n, m, N_1 and N_2 in this case will equal:

$$n = \frac{\eta_0}{k/c} + 1,\ m = 1/n \left(\frac{\eta_1}{k/c} + 1\right),\ N_1 = b/c,\ N_2 = \alpha .$$

The characteristic of $J(j\Omega) = N_1N_2 \cdot J_0(j\Omega) = (b/c) \cdot (\text{alpha}) \cdot (N/M^2) \cdot (-1)/f\Omega(1+f\Omega) = \mu \cdot (-1)/j\Omega(1+j\Omega)$ where $\mu = (b/c)(N/M^2) \cdot \alpha$ is shown in Figure 35 for $\mu = 1$ and $\mu = 10$. By assigning various values to n and m and plotting for them the impedance characteristics of the reduced nonlinear element $1/J_{NEReO}$, it is possible to determine the cricical value of μ_{cr} for which the characteristics of $1/J_{NEReO}$ and $J(j\Omega)$ do not as yet intersect and, hence, the system remains stable. Figure 35 also shows the characteristic of $1/J_{NEReO}$ for n = 3 and m = 2.

4. A Nonlinear Element Connected in Paralled With a Linear Element

To produce stability and fast action in regulation systems, parallel circuits (open-loop and feedback) are often used. In such systems a nonlinear element can be present in one of the parallel circuits and thus be connected in parallel with linear elements.

After verifying stability in all the linear circuits, it is necessary not only to (a) disconnect the system at the circuit containing the nonlinear element and (b) determine the operator admittance of the system's linear part and, after plotting the amplitude-phase characteristic, and the impedance characteristic of the nonlinear element, but also to determine, according to the above, the conditions necessary for stability.

Figure 36 gives the schematic of a regulation system with feedback in which a nonlinear element is shunted by a linear element. The equations for the system will be as follows:

$$x_3 = J_2(p)e,$$
$$x_5 = J_3(p)x_3,$$
$$x_4 = J_1(p)x_3,$$
$$x_1 = x_4 + x_5. \tag{43}$$

It follows from equation (43) that the operator admittance of the system's linear part equals

$$J_0(p) = x_1/e = J_2(p)\,[J_1(p) + J_3(p)] \tag{44}$$

The operator equation of the closed system will be

$$J_0(p) = 1/J_{NE} \tag{45}$$

As an example, let us examine the system of stabilization for the course of a neutral self-piloted airplane with a constant-speed servomotor with a zone of insensitivity [10]. The schematic is shown in Figure 37. The equations for this system have the following form:

$$\begin{aligned}
&p^2\phi + Mp\phi = -N\eta,\\
&\gamma = \phi + \beta p\phi,\\
&\psi = \gamma - \tfrac{1}{b}\eta,\\
&F(\psi) = -F(-\psi) = \begin{cases} 0 \text{ when } \psi < \psi_0, \\ 1 \text{ when } \psi > \psi_0 \end{cases}\\
&F(\psi) = \frac{1}{\psi_0} J_{NE\Pi O} \cdot \psi\\
&\eta = \frac{1}{pT_s} F.
\end{aligned} \tag{46}$$

In accordance with equations (43) and (44), we shall obtain the following expressions for the operator admittance of a system's linear part

$$J_0(p) = 1/pT_s \cdot [-N(1+\beta p)/p(p+M) - 1/b] \tag{47}$$

The operational equation of a closed system will be:

$$\frac{1}{pT_s}\left[-\frac{N(1+\beta p)}{p(p+M)}-\frac{1}{b}\right]=\psi_0/J_{NEIIO} \tag{48}$$

or

$$\frac{1}{\psi_0 pT_s}\cdot\left[-\frac{N(1+\beta p)}{p(p+M)}-\frac{1}{b}\right]=1/J_{NEIIO} \tag{48'}$$

Let us examine the left side of equation (48')

$$J_{01}(p)=\frac{1}{\psi_0 pT_s}\left[-\frac{N(1+\beta p)}{p(p+M)}-\frac{1}{b}\right] \tag{49}$$

Let us make the substitution q = p/M and also introduce the notation

$A=\beta\cdot M,\quad B=M^2/Nb,\quad C=bMT_s\cdot\psi_0$

Equation (49) will then take the form:

$$J_0(q)=\frac{1}{Cq}\left[-\frac{1+Aq}{Bq(1+q)}-1\right] \tag{49'}$$

and consequently the complex admittance (amplification factor) will be:

$$J_{01}(j\Omega)=\frac{1}{Cj\Omega}\left[-\frac{1+Aj\Omega}{Bj\Omega(1+j\Omega)}-1\right]$$
$$=X(\Omega)+jY(\Omega) \tag{49''}$$

Since the impedance characteristic of the nonlinear element, namely $1/J_{NEIIO}$, lies on the positive real axis and to the right of the point (1, j0), the condition necessary for absolute stability in the system will obviously be:

$Y(\Omega)$ greater than 0

which according to (49") will hold true for $(A-1)/B(1+\Omega^2)+1$ greater than 0.

This inequality will be satisfied for all Ω's, if

$A+B$ is greater than 1.

With this condition the amplitude-phase characteristic of $J_{01}(j\Omega)$ (for values of Ω between 0 and infinity) lies entirely in the first quadrant and the system is stable for any value of C. For $A+B$ less than 1, the amplitude-phase characteristic of $J_{01}(j\Omega)$ will have the form shown in Figure 3C.

The segment K, cut off by the characteristic on the real axis, as it follows from (49"), equals

$$K=X(\Omega_0)=(1+A\Omega_0^2)/BC\Omega_0^2(1+\Omega_0^2)$$

where the "reduced" intersection frequency is equal to

$$\Omega_0=[(1-A+B)/B]^{\frac{1}{2}}$$

and therefore

$$K = (A+B)/C(1-A+B).$$

When the nonlinear element ($J_{NEIIO} = 1$) is absent, the system will be stable for K greater than 1 that is for $(A+B)/C(1-A-B)$ greater than 1; and will be unstable for K less than 1. When the system has a nonlinear element NEII present (see Figure 5), it will be stable for K between 1 and 1.57 ($= 1/(J_{NEIIO})_{max}$). For K greater than 1, oscillations will develop with a reduced frequency:

$$\Omega_0 = \sqrt{\frac{1-A-B}{B}}$$

and with an amplitude determined by the equation:

$$K = 1/J_{NEIIO} \tag{50}$$

Equation (50) corresponds to two values of the relative amplitude of oscillations $A_1/a = \psi_{max}/\psi_0$ (see Figure 38), determined by the points M and N. The point N, in accordance with the stated criterion for the stability of oscillations, defines the unstable process; the point M, the stable one. In other words, oscillations of small amplitude are stable and those of great amplitude are unstable. As may be seen from Figure 5, the relative amplitude A_1/a of oscillations cannot be greater than 1.41.

For K less than 1 the system develops oscillations whose amplitude cannot be limited by the discussed nonlinearity.

When we compare the results obtained according to the first approximation with the exact solution of problem [10], we are convinced of their excellent qualitative agreement.

Conclusion

This work has as its purpose the investigation, in the first approximation, of the stability of tracking systems and autoregulation systems with nonlinear elements and introduces the concept of equivalent admittance and impedance of a nonlinear element in accordance with the method of "harmonic balance (equilibrium)." The characteristic of equivalent impedance is established. The problem of stability is solved by plotting the amplitude-phase characteristic of the system's linear part. If this characteristic does not contain (encompass or surround) or intersect the characteristic curve of the equivalent impedance, then the system with a nonlinear element in the closed state is stable; in the contrary case, auto-oscillations appear. Conditions necessary for stability, frequency, and amplitudes of oscillation are determined in this work.

Since in many cases, tracking systems and autoregulation systems are quasi-harmonic, the first approximation practically gives completely satisfactory results.

Many examples illustration the stated methods of investigation are analyzed.

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[Figures follow.]

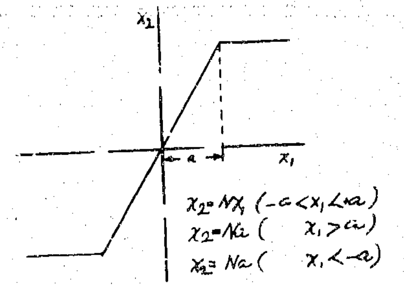

Figure 1. General Graphical Behavior of the Power Amplifying Element of the Regulator Function $x_2 = \phi(x_1)$.

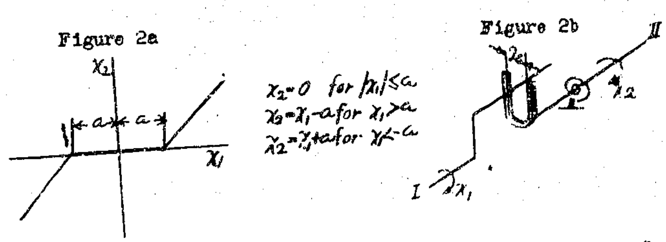

Figure 2. The Characteristic (a) and Model (b) of a Scheme of a Type-NE1 Nonlinear Element (zone of intensitivity or fork)

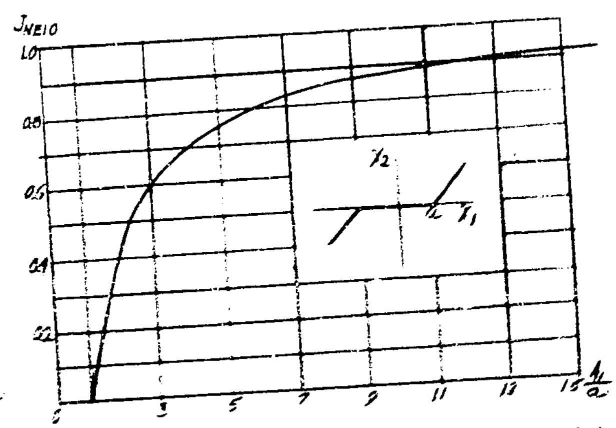

Figure 3. Graph Showing the Dependence of the Equivalent Admittance of a Nonlinear Element of Type NE1 Upon the "Amplitude Ratio" A_1/a

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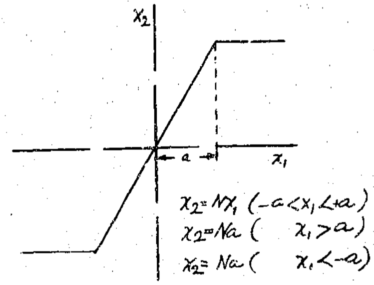

Figure 1. General Graphical Behavior of the Power Amplifying Element of the Regulator Function $x_2 = \phi(x_1)$.

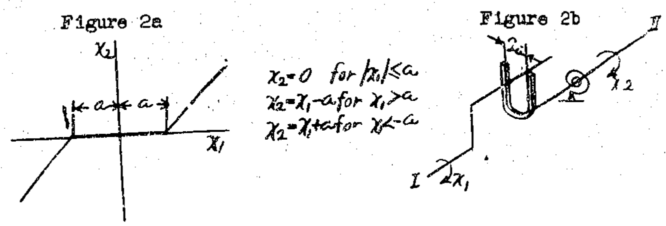

Figure 2. The Characteristic (a) and Model (b) of a Scheme of a Type-NEI Nonlinear Element (zone of insensitivity or [illegible])

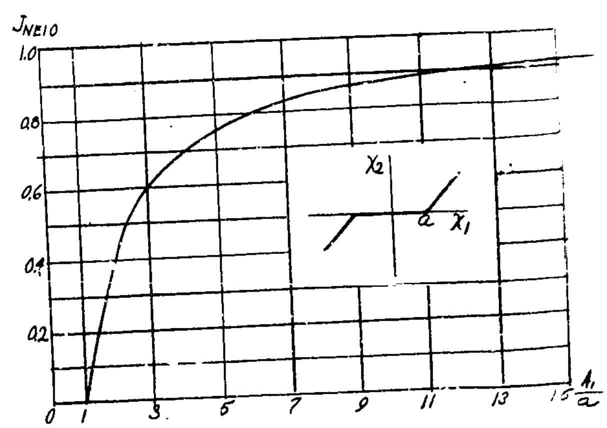

Figure 3. Graph Showing the Dependence of the Equivalent Admittance of a Nonlinear Element of Type NEI Upon the "Amplitude Ratio A_1/a

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Figure 4a

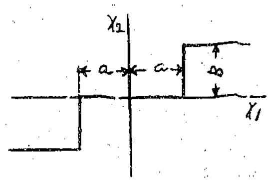


Figure 4b

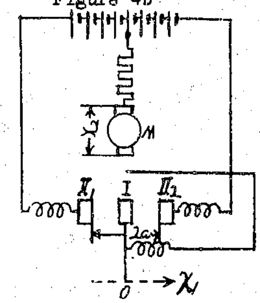


Figure 4. The Characteristic (a) and Schematic (b) of a Nonlinear Type-NEII Element (Note. "Characteristic" always means here "characteristic curve.")

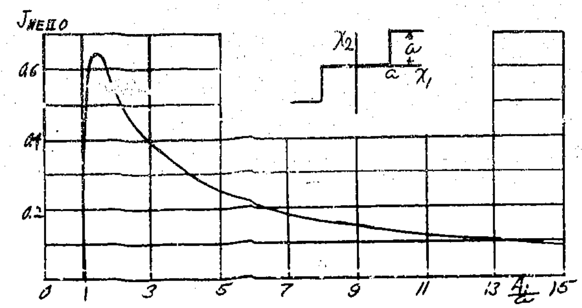
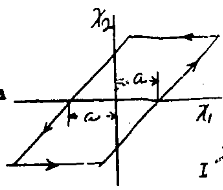
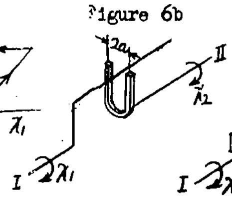


Figure 5. Graph Showing the Dependence of the Equivalent Admittance of a Nonlinear Type-NEII Element Upon the Amplitude Ratio A_1/a

Figure 6b

Figure 6a





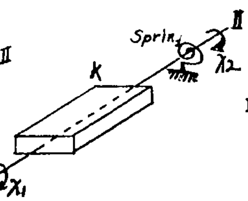


Figure 6c

Figure 6. The Characteristic (a) and Schematic (b and c) of a Nonlinear Type-NEIII Element (free play or friction)

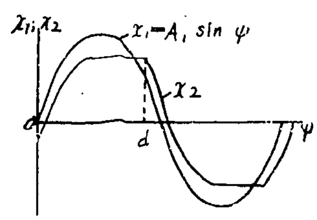


Figure 7. Graph Showing the Dependence of Input (x_1) and Output (x_2) Upon ψ

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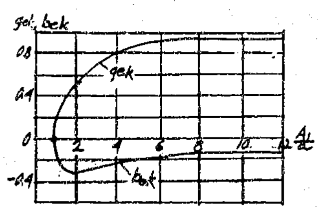

Figure 8. Graph of Active (g_{ek}) and Reactive (b_{ek}) Conductances Making up the Equivalent Admittance of a Nonlinear Type-NEIII Element Versus Relative Amplitude A_1/a

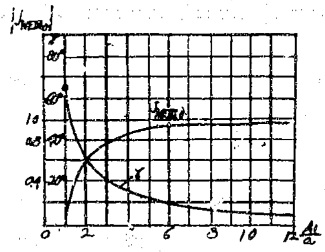

Figure 9. Graph of Modulus (Absolute Value or J_{NEIII0}) and Phase (gamma) of the Equivalent Admittance of a Nonlinear Type-NEIII Element Versus Ratio A_1/a

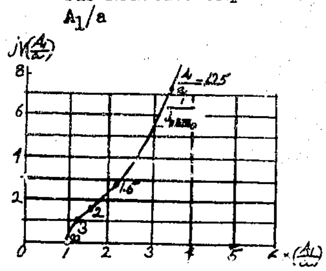

Figure 10. Characteristic of the Equivalent Impedance of a Nonlinear Type-NEIII Element

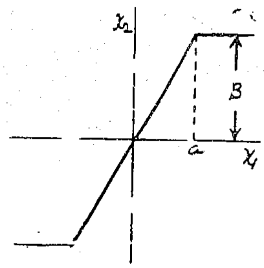

Figure 11. Characteristic of a Nonlinear Type-NEIV Element

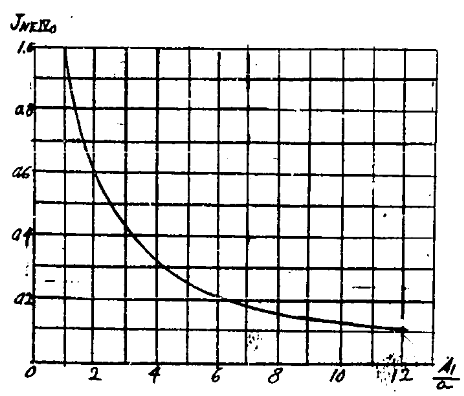

Figure 12. Graph of Equivalent Admittance of a Nonlinear Type-NEIV Element Versus the Ratio A_1/a

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Figure 13a Figure 13b

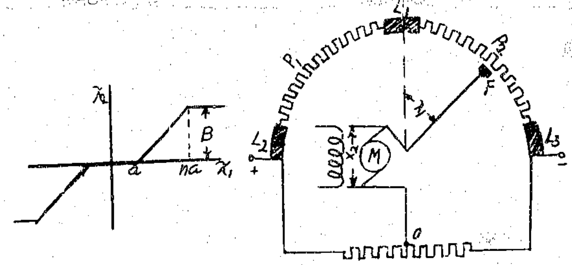


Figure 13. Characteristic and Schematic of a Nonlinear Type-NEV Element

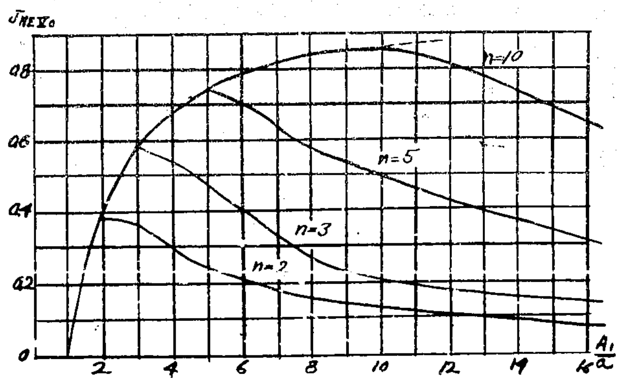


Figure 14. Graph of the Dependence of the Equivalent Admittance of a Nonlinear Element of Type-NEV Upon the Ratio A_1/a for Different Values of n

Figure 15a Figure 15b

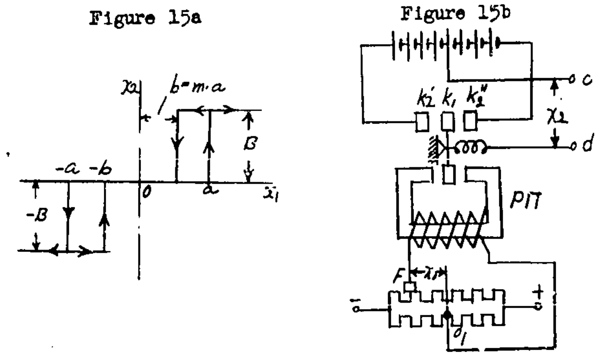


Figure 15. Characteristic and Schematic of a Nonlinear Type-NEVI Element

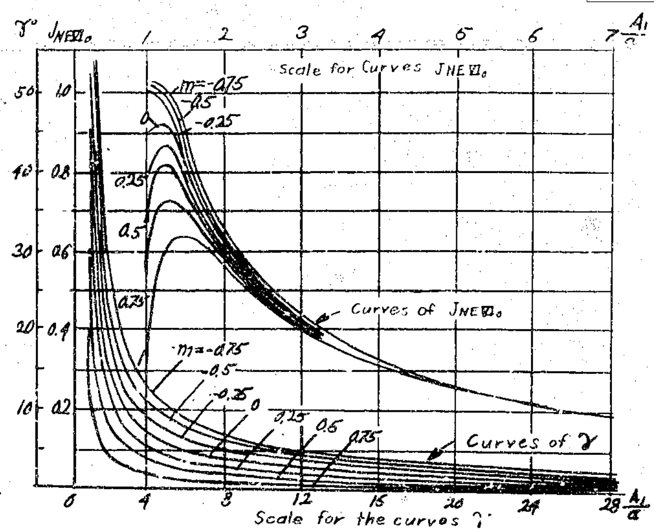


Figure 16. Graph of Modulus (J_{NEVIO}) and Phase (gamma) of the Equivalent Admittance of a Nonlinear Type-NEVI Element Versus the Ratio A_1/a

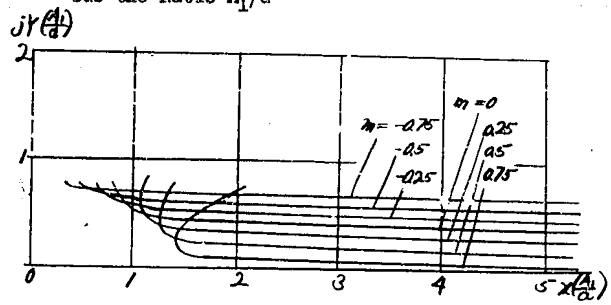


Figure 17. Characteristic of the Equivalent Impedance of a Nonlinear Type-NEVI Element

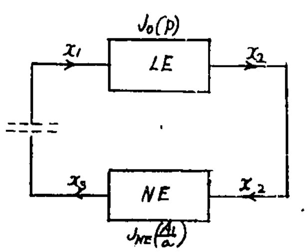


Figure 18. A Schematic of a Regulation System (Note: LE -- linear; NE -- nonlinear)

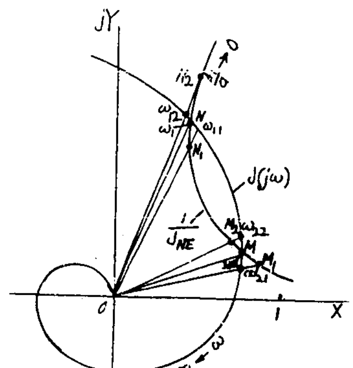


Figure 19. The Amplitude-Phase Characteristic of a Linear Element $J(j\omega)$ and Impedance Characteristic of a Nonlinear Element $1/J_{NE}$

Figure 20a

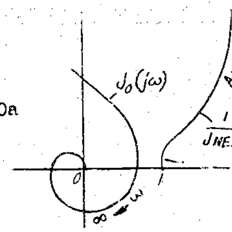


Figure 20b

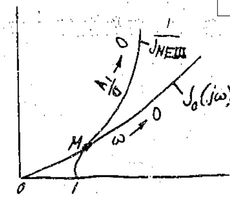


Figure 20c

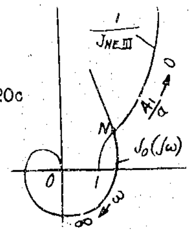


Figure 20d

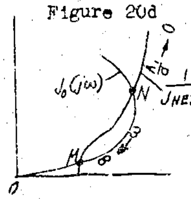


Figure 20e

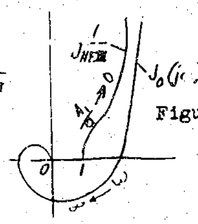
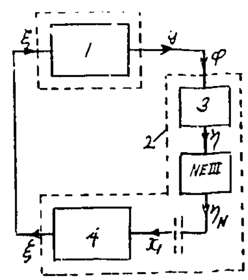


Figure 20. Various Cases of Common (simultaneous) Disposition of the Characteristics of Linear and Nonlinear Elements: a-Absolutely Stable System; b-Unstable System; c-Conditionally Stable System; d-Conditionally Stable System; e-Absolutely Unstable System

Figure 21a



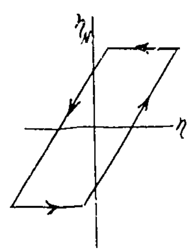


Figure 21b

Figure 21. a-Schematic of the Primary Regulation of an Inertia Object; 1- Object of Regulation, 2- Regulator, 3- Measuring Element, 4- Servomotor. b-Characteristic of a Nonlinear Type-NEIII Element

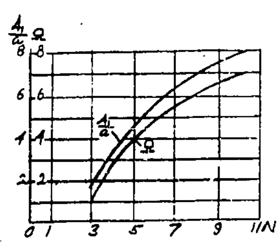


Figure 22. Characteristics of the Linear and Nonlinear Elements in a Primary Regulation Scheme of an Inertia Object

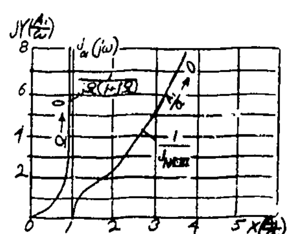


Figure 23. The Ratio A/a and Reduced Frequency Ω of Oscillation Versus Amplification Factor N

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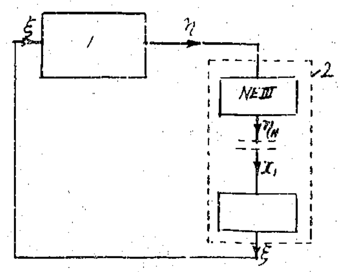

Figure 24. Schematic Showing Regulation by Means of a Regulator Having Mass and Viscous and Dry Frictions: 1- Object of Regulation and 2- Regulator

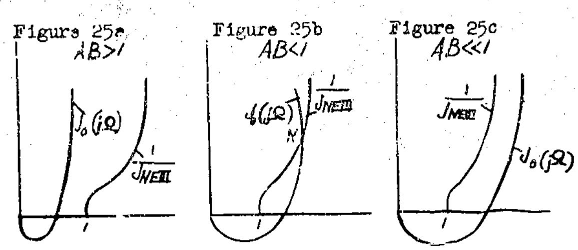

Figure 25. The Characteristic Curves of the Linear and Nonlinear Elements of a Regulation System That Is Controlled by a Regulator Possessing Mass and Viscous and Dry Frictions. a-Absolutely Stable; b-Conditionally Stable; c-Absolutely Unstable

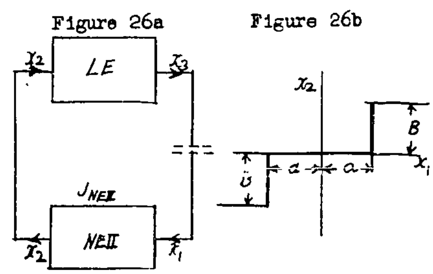

Figure 26. a-Schematic of a Regulation With a Nonlinear Type-NEII Element; b-Characteristic of a Nonlinear Type-NEII Element

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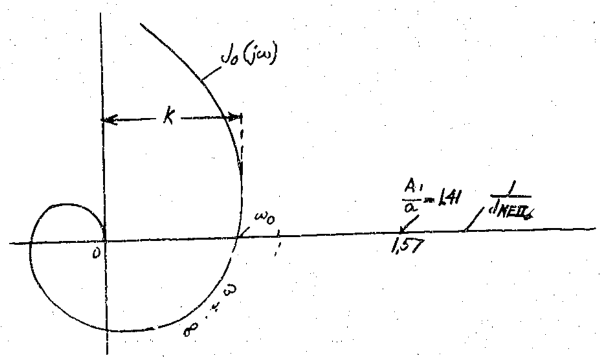

Figure 27. The Amplitude-Phase Characteristic of the Linear Part of the System $J_0(j\omega)$ and the Impedance Characteristic of the Nonlinear Element $1/J_{NEII0}$

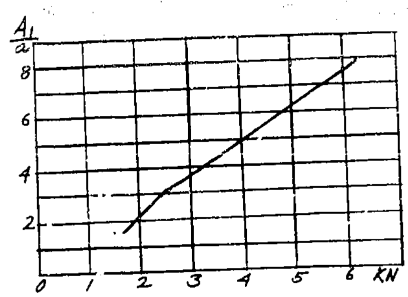

Figure 28. Dependence of the Ratio A_1/a of Auto-Oscillation Upon Amplification Factor kN

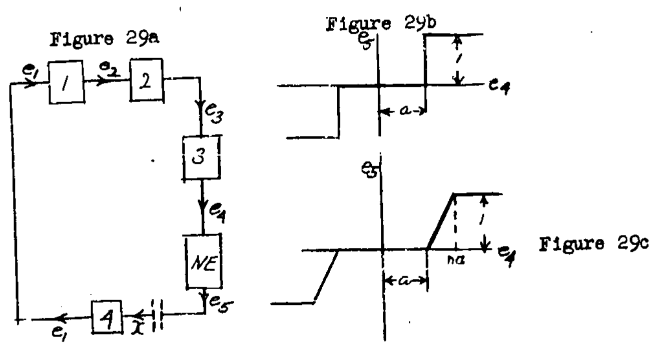

Figure 29. a-The Schematic of Voltage Regulation With a Nonlinear Element: 1- Exciter, 2- Generator, 3- Regulator's Measuring Element, 4- Servomotor; b-Characteristic of the Nonlinear Element of the Schematic With Zone of Insensitivity; c-Characteristic of the Nonlinear Element of the Schematic in the Case Where the Servomotor Has a Variable Velocity With a Zone of Insensitivity

Figure 30a

Figure 30b1

Figure 30b2

Figure 30c

Figure 30d

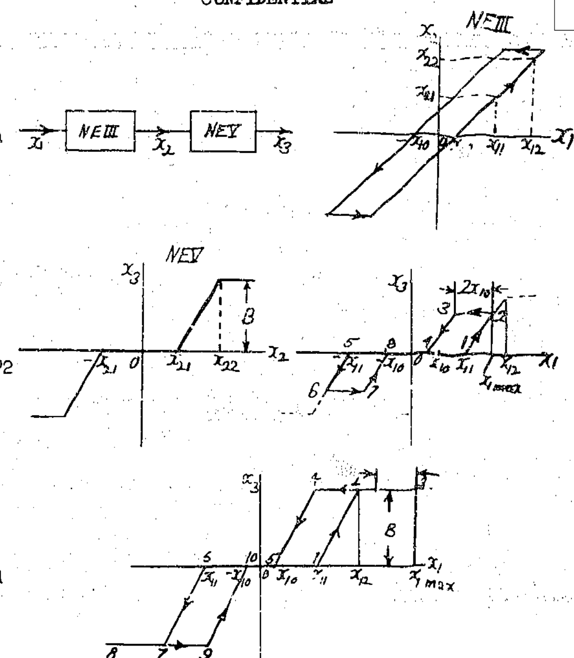

Figure 30. Structure of the Characteristic of the Reduced (lumped or resultant) Nonlinear Element With Respect to the Given Characteristics of the Component Nonlinear Elements: a-Schematic of Two Nonlinear Elements Connected in Series; b-Characteristic of These Elements; c-Characteristic of the Reduced Nonlinear Element $x_3(x_1)$ for x_1 Less Than x_{1max} Less Than x_{12}; d-Characteristic of the Reduced Nonlinear Element $x_3(x_1)$ for x_{1max} Greater Than x_{12}

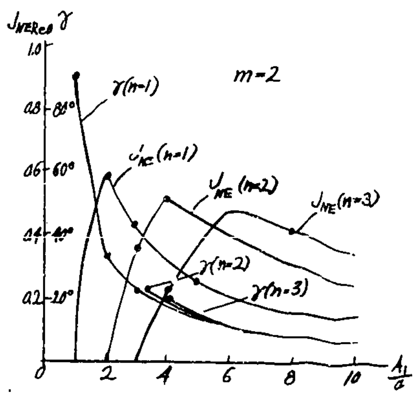

Figure 31. Graph of the Modulus and Phase of the Equivalent Admittance of the Reduced Nonlinear Element (type NEIII + NEV) Versus the Relative Amplitude A_1/a

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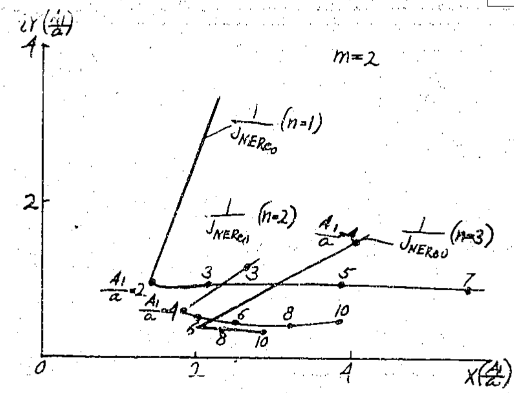

Figure 32. Characteristics of the Equivalent Impedance of the Reduced Nonlinear Element of Type (NEIII+NEV)

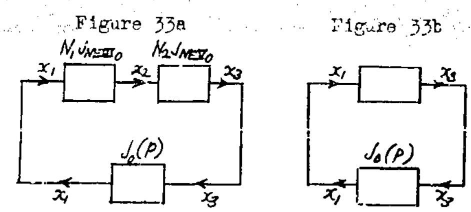

Figure 33. a-Simplified Schematic of a Regulation System With Two Nonlinear Elements; b-Equivalent System of Figure a

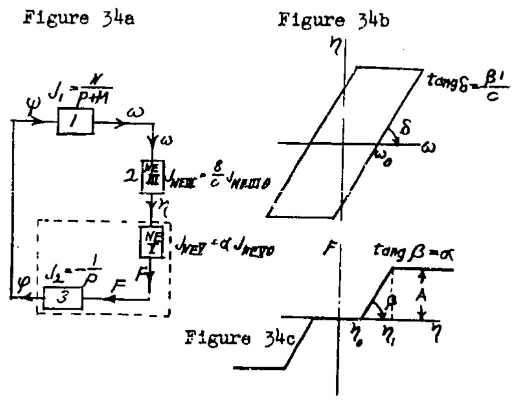

Figure 34. a-Simplified Schematic of a System for Regulating the Revolutions of an Aircraft Engine: 1- Regulated Object; 2- Indicator; 3- Servomotor: b-Characteristic of the Nonlinear Element NEIII; c-Characteristic of the Nonlinear Element NEV

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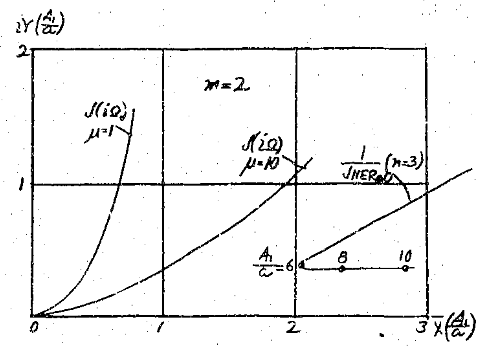


Figure 35. Amplitude-Phase Characteristics of a Linear System and Impedance Characteristic of a Nonlinear Element

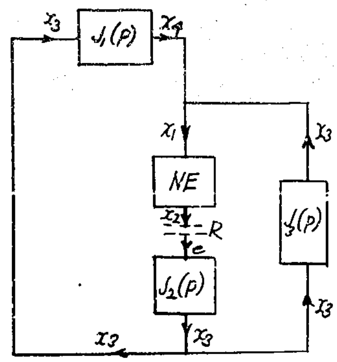


Figure 36. A Schematic in Which a Nonlinear Element Is Shunted by a Linear Element

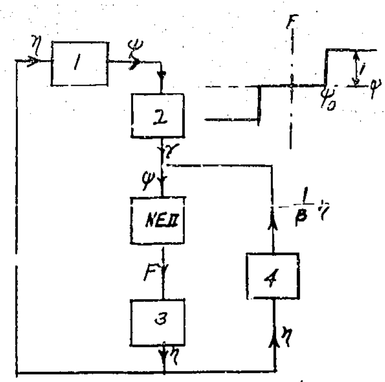
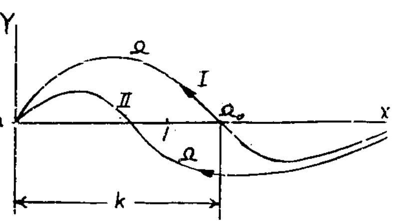


Figure 37. Simplified Schematic for the Regulation of an Airplane's Course: 1- Airplane; 2- Measuring Element; 3- Servomotor; 4- Feedback

Figure 38a



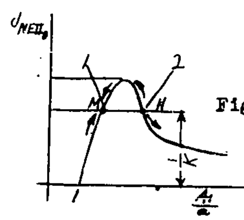


Figure 38b

Figure 38. a-Amplitude-Phase Characteristics of a System for Regulating an Airplane's Course: I-The Linear Part of the System Is Stable; II-The Linear Part of the System is Unstable; b-Graph to Determine the Amplitude of Auto-Oscillation: 1- Point Corresponding to a Stable State; 2- Point Corresponding to an Unstable State

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